Parametric Subharmonic Instability of Internal Waves: Locally Confined Beams vs. Monochromatic Wavetrains

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Executive Summary

Internal gravity wavetrains in continuously stratified fluids are generally unstable as a result of resonant triad interactions that amplify short-scale perturbations with frequency equal to one half of that of the underlying wave. This so-called parametric subharmonic instability (PSI) has been studied extensively for spatially and temporally monochromatic waves, as a potential mechanism for transferring energy from large-scale internal waves to small-scale mixing in oceans. However, the PSI found in stability analyses of purely monochromatic wavetrains may not be entirely relevant to ocean internal waves, as such highly idealized disturbances are not encountered in the field. Instead, tidally generated ocean internal waves are often in the form of beams—time-harmonic plane waves with general spatial profile—that propagate along a direction to the vertical determined by the wave frequency. Here, we present an asymptotic analysis of PSI of internal wave beams in an effort to understand how these more realistic disturbances differ, in regard to PSI, from monochromatic waves.

Our analysis assumes a weakly nonlinear wave beam of $O(\varepsilon)$ amplitude ($\varepsilon << 1$) and $O(1)$ width, to which are superposed short-scale ($O(\varepsilon^{1/2})$ wavelength) subharmonic perturbations (frequency ½ of the beam frequency), as sketched in Fig. 1. These scales ensure that nonlinear effects due to resonant triad interactions balance with dispersive effects due to the modulation of the subharmonic perturbations by the underlying beam. For beams with general localized profile, it is found that triad interactions are not strong enough to bring about instability in the limited time that the subharmonic perturbations, which propagate with their respective group velocity, overlap with the beam. On the other hand, for quasi-monochromatic wave beams whose profile comprises a sinusoidal carrier modulated by a locally confined envelope, PSI is possible if the beam is wide enough; specifically, if it comprises at least $O(\varepsilon^{-1/2})$ carrier wavelengths. An important

exception arises when the beam frequency is nearly twice the inertial frequency due to background rotation; under this condition, PSI is possible for beams of general locally confined profile, as subharmonic perturbations of near-inertial frequency have small group velocity and stay in contact with the underlying beam longer, thus extracting more energy, than their non-inertial counterparts. The predictions of the asymptotic theory are in keeping with numerical simulations.

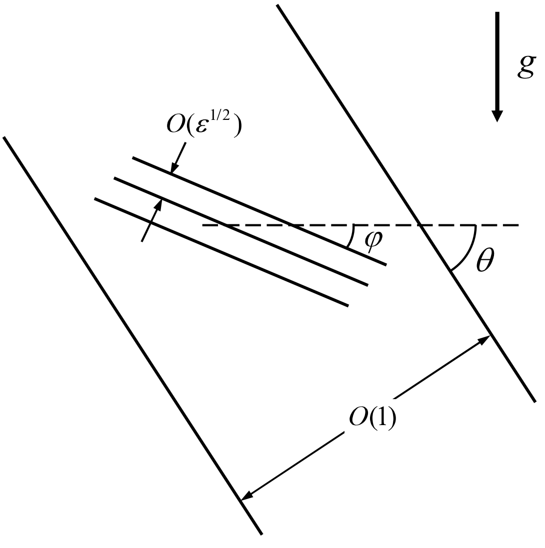

Figure 1. Geometry of beam—perturbation interaction. The underlying wave beam of $O(\varepsilon)$ amplitude and $O(1)$ width has frequency ω and propagates at an angle θ to the horizontal such that $\omega = \sin\theta$. Subharmonic perturbations are short-crested ($O(\varepsilon^{1/2})$ wavelength) nearly monochromatic wave packets with frequency close to $\omega/2$ that propagate at an angle φ to the horizontal, with $\sin\varphi = (\sin\theta)/2$.

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